13.3, Part One) Arc Length

1. Review of concepts from Calculus I, especially the differentials of x and y

Suppose we have a function $y = f(x)$. Let a be a particular value of x. The corresponding value of y is $f(a)$, and the point $(a,f(a))$ lies on the graph of the function; call this point P. Let L be the tangent line to the graph of the function at the point P. In Calculus I, we learned how the tangent line is obtained by applying the concept of the *limit*. Let b be a value of x nearby a. The corresponding value of y is $f(b)$, and the point $(b,f(b))$ lies on the graph of the function; call this point Q. The line passing through the points P and Q is known as a **secant line** (the word *secant* means "to cut," and the line *cuts through* the graph of the function at the points P and Q). If we let b approach a, then the point Q approaches the point P, and the secant line adjusts accordingly. The closer b comes to a, the closer Q comes to P, and the closer the secant line comes to the tangent line. We say that the tangent line is the *limiting position* of the secant line as b approaches a. In other words, the *limit* of the secant line as b approaches a is the tangent line.

Likewise, the *slope* of the tangent line is the limit of the *slope* of the secant line as b approaches a. The slope of the secant line is $\frac{f(b)-f(a)}{b-a}$, so the slope of the tangent line is $\lim_{b \to a} \frac{f(b)-f(a)}{b-a}$.

Let $\Delta x = b - a$. So $b = a + \Delta x$. The slope of the secant line may be expressed as $\frac{f(a+\Delta x)-f(a)}{\Delta x}$. When b approaches a, Δx approaches zero. Hence the slope of the tangent line may be expressed as $\lim_{\Delta x \to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x}$. Of course, this is the derivative of the function at $x = a$, i.e., $f'(a)$.

The derivative of the function may be expressed in **Leibniz notation** as $\frac{dy}{dx}$. On the one hand, this is merely a synonym for $f'(x)$. On the other hand, we may think of it as the literal quotient of two quantities, dy and dx, which are known as **differentials** (they are referred to as the **differential of** y and the **differential of** x, respectively). These quantities have a very meaningful geometric interpretation. In the context of the above discussion, where we are examining the tangent line at the point $(a,f(a))$, the differentials would be interpreted as follows...

dx represents a nonzero *deviation* of x from a. dx may be positive or negative. *Positive* dx means deviation to the *right* of a, which gives us an x value *greater* than a. *Negative* dx means deviation to the *left* of a, which gives us an x value *less* than a. In either case, the value of x we obtain from this deviation is expressed as $a + dx$.

Now think of the tangent line as the graph of a linear function, $L(x)$, known as the **linearization** of f at $x = a$. The point P lies on the graph of $L(x)$, so $L(a) = f(a)$. Since the tangent line passes through the point $(a,f(a))$ and has slope $f'(a)$, we may use the point-slope formula to write its equation, giving us $y - f(a) = f'(a)(x-a)$, or $y = f'(a)(x-a) + f(a)$. Consequently, $L(x) = f'(a)(x-a) + f(a)$.

When $x = a + dx$, the value of the linearization is
$L(a + dx) = f'(a)(a + dx - a) + f(a) = f'(a)dx + f(a)$.

When x varies from a to $a + dx$, $L(x)$ varies from $L(a)$ to $L(a + dx)$, and hence the *change* in $L(x)$ is $L(a + dx) - L(a)$, which equals $f'(a)dx + f(a) - f(a)$, which simplifies to $f'(a)dx$. This is precisely the meaning of dy. In other words, by definition, $dy = f'(a)dx$, i.e., dy *means* the change in $L(x)$ when x varies from a by the amount dx. Since dx is nonzero, we may divide both sides of this equation by dx, giving us $\frac{dy}{dx} = f'(a)$.

dy can be positive or negative of zero. *Positive* dy means that the value of $L(a + dx)$ is *greater* than $L(a)$. *Negative* dy means that the value of $L(a + dx)$ is *less* than $L(a)$. *Zero* dy means that the value of $L(a + dx)$ is *equal* than $L(a)$. The third case occurs when the tangent line is *horizontal*. The first and second cases occur when the tangent line is *oblique* (i.e., either slanting up or down). To be more specific, the first case, positive dy, occurs either when the tangent line slants upward and dx is positive, or when the tangent line slants downward and dx is negative. The second case, negative dy, occurs either when the tangent line slants upward and dx is negative, or when the tangent line slants downward and dx is positive.

When x varies from a to $a + dx$, $f(x)$ varies from $f(a)$ to $f(a + dx)$, and hence the *change* in $f(x)$ is $f(a + dx) - f(a)$, which we denote Δf.

The purpose of dy is to *approximate* Δf. This is an extension of the idea that the function $L(x)$ is an *approximation* of the function $f(x)$.

When dx is small, these approximations, i.e., $dy \approx \Delta f$ and $L(x) \approx f(x)$, are quite good.

For example, let $f(x) = x^2$ and let $a = 3$. The tangent line has slope 6, so $L(x) = 6x - 9$. Let $dx = -0.07$, so $a + dx = 2.93$. $f(3) = 9$, and $f(2.93) = 8.5849$, so $\Delta f = -0.4151$. In comparison, $dy = (6)(-0.07) = -0.42$. Note that $L(2.93) = 8.58$, which is very close to $f(2.93)$.

Most mathematical concepts have more than one purpose or use, so when I say "the purpose of dy," what I mean is its *most basic* purpose. It certainly has other uses!

2. The Differential of Arc Length, ds

Suppose dy is nonzero (which means our tangent line is either upward sloping or downward sloping, not horizontal). Let M be the point $(a + dx, L(a + dx))$, which lies on the graph of $L(x)$. As before, let P be the point $(a, f(a))$, which also lies on the graph of $L(x)$. The line segment $\overline{PM}$ is neither horizontal nor vertical. Let us construct a right triangle with this line segment as its hypotenuse. Let N be the point $(a + dx, f(a))$ (in other words, N is the point that aligns vertically with M and horizontally with P). So triangle MNP is a right triangle whose right angle is $\measuredangle MNP$. The hypotenuse is $\overline{PM}$, the horizontal leg is $\overline{PN}$, and the vertical leg is $\overline{MN}$.

The length of the horizontal leg is $|dx|$, and the length of the vertical leg is $|dy|$. (We must use absolute values because length must be positive, whereas each differential could possibly be negative.)

By the Pythagorean Theorem, the length of the hypotenuse is $\sqrt{dx^2 + dy^2}$. We define this as a new differential, ds. In other words, by definition, $ds = \sqrt{dx^2 + dy^2}$. Note that $dx^2 + dy^2 = ds^2$. For reasons to be made clear shortly, ds is known as the **differential of arc length**.

Whereas dx and dy have the potential of being negative, ds cannot be negative, since it is defined as a principal square root, which is never negative.

By definition, dx cannot be zero, but dy can be zero (this occurs when the tangent line is horizontal). Our discussion of the right triangle MNP was predicated on the assumption that dy was not zero. However, our algebraic definition of ds makes perfectly good sense even when dy is zero. When $dy = 0$, we have $ds = \sqrt{dx^2 + 0^2} = \sqrt{dx^2} = |dx|$.

Since dx cannot be zero, ds must always be positive.

So far, we have given the definition of ds, and discussed its geometric interpretation when dy is nonzero. Now let's discuss the purpose of ds.

Recall that $dy \approx \Delta f = f(a+dx) - f(a)$. Let K be the point $(a+dx, f(a+dx))$, which lies on the graph of $f(x)$. K aligns vertically with M.

The graph of $f(x)$ is a curve. When we restrict x to the closed interval $[a, a+dx]$ (when dx is positive) or $[a+dx, a]$ (when dx is negative), we obtain an **arc** of the curve having endpoints P and K. The measurement of this arc is known as its **arc length**, and may be denoted $arclength(P,K)$.

When dx is small, M lies close to K, so $arclength(P,K)$ is closely approximated by the length of the line segment $\overline{PM}$, which is precisely ds. So $ds \approx arclength(P,K)$.

For example, consider the function $f(x) = \sqrt{25 - x^2}$, which has domain $[-5, 5]$. The graph of this function is the top half of a circle with radius five centered at the origin. Let $a = 4$, so $f(a) = f(4) = 3$ and P is the point $(4,3)$. $f'(x) = -x(25-x^2)^{-1/2}$, so $f'(a) = -\frac{4}{3}$, and $L(x) = -\frac{4}{3}x + \frac{25}{3}$. Let $dx = -1$, so $a + dx = 3$, $f(a+dx) = f(3) = 4$, and K is the point $(3,4)$. $L(a+dx) = L(3) = \frac{13}{3}$, so M is the point $(3, \frac{13}{3})$ or $(3, 4\frac{1}{3})$.

For the moment, let's leave calculus aside, and just use trigonometry. The points P and K lie on the circle centered at the origin, which we will refer to as the point O. So the radian measure of $\triangle POK$ is $\arctan\frac{4}{3} - \arctan\frac{3}{4} \approx 0.284$. Therefore $arclength(P,K) \approx 1.42$ (the length of an arc of a circle is equal to the radius multiplied by the radian measure of the subtended angle, and $5 \cdot 0.284 = 1.42$). On the other hand, we can compute the length of line segment $\overline{PM}$ using the distance formula; we get $\sqrt{(4-3)^2 + (4\frac{1}{3} - 3)^2} = 1.25$.

Now let's bring back our calculus. Since $dx = 1$, $dy = f'(a) = -\frac{3}{4}$, so $ds = \sqrt{1^2 + (-\frac{3}{4})^2} = \sqrt{\frac{25}{16}} = \frac{5}{4} = 1.25$.

In this example, the differential of arc length gave us 1.25 as an approximation to the true arc length, 1.42. It wasn't a very accurate approximation. That's because we were using a relatively large dx, namely, -1. We chose this value to allow for simple calculations. In reality, to obtain a good approximation of arc length, we would use a smaller dx, such as -0.1 or -0.01.

In the above example, we were able to compute the true arc length using trigonometry, since we were dealing with an arc of a circle. Usually we can't do this, because usually we are dealing with noncircular curves.

3. Dealing with ds in the case of a parametrically defined curve

Suppose we have a curve defined by the parametric equations $x = x(t)$, $y = y(t)$. So $\frac{dx}{dt} = x'(t)$ and $\frac{dy}{dt} = y'(t)$. Therefore $dx = x'(t)dt$ and $dy = y'(t)dt$.

By substitution, $ds = \sqrt{dx^2 + dy^2} = \sqrt{(x'(t)dt)^2 + (y'(t)dt)^2} = \sqrt{x'(t)^2 dt^2 + y'(t)^2 dt^2} = \sqrt{(x'(t)^2 + y'(t)^2)dt^2} = \sqrt{x'(t)^2 + y'(t)^2}\sqrt{dt^2} = \sqrt{x'(t)^2 + y'(t)^2}\,|dt|$

Since t represents time, and we generally think of time as moving *forward*, it is reasonable to adopt the convention that dt is always positive. Thus, we may drop the absolute value, which gives us the equation $ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt$.

Let's pause to think about what dt represents. Suppose α is a value of t such that $x(\alpha) = a$. Just as dx represents the deviation of x from a, likewise dt represents the deviation of t from α. Although dt is necessarily positive, dx could be negative, because a particle moving along a curve could be moving leftward as time goes forward.

When we have $y = f(x)$ with no parameterization involved, x is the independent variable and y is the dependent variable (depending on x). Under these circumstances, dx is likewise an independent variable, and dy is a dependent variable (when the point of tangency has been specified, dy is a function of dx). Once we have introduced a parameterization, so that $x = x(t)$ and $y = y(t)$, we now have the parameter t as our independent variable, and now x and y are both dependent variables (depending on t). Under these circumstances, dt is likewise an independent variable, and dx and dy are both dependent variables (when the point of tangency has been specified–i.e., when $t = \alpha$ has been specified–then dx and dy are both functions of dt).

Earlier, we discussed how dy serves as an approximation to Δf when we are dealing with a function $y = f(x)$. We may write Δy in place of Δx. Thus, dy serves as an approximation to Δy.

Similarly, once we have introduced our parameterization of x and y as functions of t, then dx serves as an approximation to Δx and dy serves as an approximation to Δy.

In our previous example, we considered the function $f(x) = \sqrt{25 - x^2}$, whose graph is the top half of a circle. Let us now widen our viewpoint to consider the complete circle, $x^2 + y^2 = 25$. It may be parameterized as $x = 5\cos t,\ y = 5\sin t,$ where $t \in [0, 2\pi]$. Let $\alpha = \arctan\frac{3}{4} \approx 0.644$. $x(\alpha) = 4$ and $y(\alpha) = 3,$ so our moving particle is located at the point $(4,3)$ when $t = \alpha$.

Let $\beta = \arctan\frac{4}{3} \approx 0.927$. $x(\beta) = 3$ and $y(\beta) = 4,$ so our moving particle is located at the point $(3,4)$ when $t = \beta$.

When time moves forward from α to β, we have $dt = \beta - \alpha \approx 0.283$. $\Delta x = 3 - 4 = -1,$ and $\Delta y = 4 - 3 = 1$. (We have negative Δx and positive Δy because the particle has moved leftward and upward.) Since $x'(t) = -5\sin t$ and $y'(t) = 5\cos t,$ $x'(\alpha) = -5(\frac{3}{5}) = -3$ and $y'(\alpha) = 5(\frac{4}{5}) = 4,$ so $dx = -3dt$ and $dy = 4dt$. Evaluating these when $dt = 0.283,$ we get $dx = -0.849$ and $dy = 1.132$. These are reasonably good approximations to Δx and Δy. We would, of course, get better approximations if we used a smaller dt.

Before we go on, let's just take a moment to verify that the equation $ds = \sqrt{x'(t)^2 + y'(t)^2}\,dt$ gives us a correct result in the above situation. Since $x'(t) = -5\sin t$ and $y'(t) = 5\cos t,$ we get $\sqrt{(-5\sin t)^2 + (5\cos t)^2}\,dt = \sqrt{25\sin^2 t + 25\cos^2 t}\,dt = 5dt$. Substituting $dt = \beta - \alpha,$ we get $ds = 5(\beta - \alpha) \approx 5(0.283) = 1.415,$ which is very close to the true arc length, 1.42.

Notice that in this analysis, we obtained a different (and far more accurate!) value of ds than we did earlier, before we introduced our parameterization. Earlier, we had chosen dx to be $-1,$ which is a fairly large value, and this gave us a relatively poor estimation of the arc length. This time, our independent variable was $dt,$ and we chose it to be $\beta - \alpha \approx 0.283,$ which is a fairly small value, and this gave us a much better estimation of the arc length.

If motion along the curve is represented by the position function $\mathbf{r}(t) = \langle x(t), y(t) \rangle,$ then the velocity is $\mathbf{v}(t) = \langle x'(t), y'(t) \rangle$ and the speed is $v(t) = |\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2}$. Thus, $ds = v(t)dt$.

4. Finding the exact value of arc length

As we have discussed, ds gives us an approximation to arc length. But how can we find the exact measure of arc length? In the above example, we were able to find the exact value using trigonometry, because we were dealing with an arc of a circle. But what if we are dealing with a curve that is *not* a circle?

The answer is quite simple, provided the curve is defined parametrically–i.e., $x = x(t)$ and $y = y(t)$.

Let α and β be two values of t, with $\alpha < \beta$, and let P and K be the corresponding points on the curve–i.e., $P = (x(\alpha), y(\alpha))$ and $K = (x(\beta), y(\beta))$. As time moves forward from α to β, our particle moves along the curve from P to K.

The equation $ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$ or $ds = v(t)dt$ gives us ds in terms of t and dt. To find $arclength(P,K)$, we simply integrate ds over the interval $[\alpha, \beta]$. In other words, $arclength(P,K) = \int_\alpha^\beta ds = \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_\alpha^\beta v(t)dt.$

For example, consider the parabola $y = x^2$. Suppose we want to find the length of the arc of this parabola from the point $(0,0)$ to the point $(3,9)$. We adopt the parameterization $x = t$, $y = t^2$. $t = 0$ gives us the point $(0,0)$ and $t = 3$ gives us the point $(3,9)$.

$x'(t) = 1$ and $y'(t) = 2t$, so $ds = \sqrt{1 + 4t^2}\, dt$. Consequently, the arclength is $\int_0^3 \sqrt{1 + 4t^2}\, dt$. To evaluate this integral, we use trigonometric substitution followed by integration by parts. We obtain $\left[\frac{1}{4} \ln\left(2t + \sqrt{1 + 4t^2}\right) + \frac{1}{2} t\sqrt{1 + 4t^2} \right]_0^3 = \frac{1}{4} \ln(6 + \sqrt{37}) + \frac{3}{2}\sqrt{37}$.

Evaluating on a calculator, we get 9.747.

We could have estimated this by computing ds, using $t = 0$ and $dt = 3$. Then $ds = 3$. This is a *terrible* estimate! But that shouldn't be surprising, since 3 is a *huge* value for dt.

5. The Arc Length Function

The formula $\int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2}\, dt$ or $\int_\alpha^\beta v(t)dt$ expresses arc length as a definite integral. As we know, the variable of integration can be replaced with a "dummy variable" and it does not alter the result, so we could also write this integral as $\int_\alpha^\beta \sqrt{x'(u)^2 + y'(u)^2}\, du$ or $\int_\alpha^\beta v(u)du$.

Say we have a curve defined parametrically by the equations $x = x(t)$, $y = y(t)$, and we have a point P corresponding to $t = \alpha$, i.e., $P = (x(\alpha), y(\alpha))$. For any value of $t > \alpha$, let K_t be the corresponding point on the curve, i.e., $K_t = (x(t), y(t))$. Then the length of the arc of the curve from P to K_t is given by the definite integral $\int_\alpha^t \sqrt{x'(u)^2 + y'(u)^2}\, du$ or $\int_\alpha^t v(u)du$. When t is allowed to vary (while α is held fixed), this integral gives us a *function* of t, which we denote $s(t)$ and which we name the **arc length function**. Its domain is $[\alpha, \infty)$, assuming the curve is defined for all $t > \alpha$.

Thus, $s(t) = \int_\alpha^t \sqrt{x'(u)^2 + y'(u)^2}\, du = \int_\alpha^t v(u)du$

$s'(t) = \frac{d}{dt}s(t) = \frac{d}{dt}\int_\alpha^t v(u)du = v(t)$, using the Fundamental Theorem of Calculus.

Consequently, the rate of change of the arc length function with respect to time is equal to the speed of motion.

If we write $s'(t)$ in Leibniz notation, we have $\frac{ds}{dt} = \sqrt{x'(t)^2 + y'(t)^2}$. Multiplying both sides of the equation by dt gives us the equation $ds = \sqrt{x'(t)^2 + y'(t)^2}\, dt$, which is the same result we obtained earlier.

In the case of motion along the parabola $y = x^2$, using the parameterization $x = t$, $y = t^2$ and letting $\alpha = 0$, we obtain the arc length function $s(t) = \int_0^t \sqrt{1 + 4u^2}\, du =$

$\left[\frac{1}{4}\ln\left(2u + \sqrt{1 + 4u^2}\right) + \frac{1}{2}u\sqrt{1 + 4u^2} \right]_0^t = \frac{1}{4}\ln\left(2t + \sqrt{1 + 4t^2}\right) + \frac{1}{2}t\sqrt{1 + 4t^2}$.

If we wish to find the length of the arc of this parabola from the point $(0, 0)$ to the point $(1, 1)$, we simply calculate $s(1)$, which is $\frac{1}{4}\ln\left(2 + \sqrt{5}\right) + \frac{1}{2}\sqrt{5}$, or about 1.479.

Let us return to the example the $x = 5\cos t$, $y = 5\sin t$. We shall now allow t to vary over the interval $[0, \infty)$. As it does so, our particle starts at the point $(5, 0)$ and then movies counterclockwise around the circle $x^2 + y^2 = 25$ infinitely many times. (It completes its first cycle when $t = 2\pi$, its second cycle when $t = 4\pi$, and so on.) The particle is moving with a constant speed that is equal to the radius, i.e., $v(t) = 5$. At any time $t \geq 0$, the distance our particle has travelled since $t = 0$ is given by the arc length function $s(t) = \int_0^t 5du = 5t$. For instance, when $t = 2\pi$, the particle has travelled a distance of 10π, which is the circumference of the circle. When $t = 6\pi$, the particle has travelled a distance of 30π, which is three times the circumference; this makes sense, because at this point in time the particle has just completed its third cycle.

If we used the parameterization $x = 5\cos(\frac{\pi}{30}t),\ y = 5\sin(\frac{\pi}{30}t)$, the speed of the particle would now be $\frac{\pi}{6}$, so $s(t) = \int_0^t \frac{\pi}{6}du = \frac{\pi}{6}t$. With this parameterization, the particle completes one cycle every 60 seconds (or one minute), so $s(60) = 10\pi$, which is the circumference; $s(15) = 2.5\pi$, which is one quarter the circumference (travelled in one quarter of a minute); and so on.

6. The Arc Length Parameter

Recall that there are many ways to parameterize a given curve. Let s be the arc length along a curve from a designated starting point P. Instead of parameterizing the curve with respect to t and then thinking of s as a function of t, as we did above, we could simply treat s as an independent parameter in its own right. We refer to it as the **arc length parameter**. We may then parameterize the curve with respect to the parameter s. In other words, we may write the parametric equations $x = x(s),\ y = y(s)$.

Consider the example $x = 5\cos t,\ y = 5\sin t$. Since $s = 5t$, we obtain $t = \frac{1}{5}s$, so the parameterization with respect to arc length is $x = 5\cos(\frac{1}{5}s),\ y = 5\sin(\frac{1}{5}s)$.

On the other hand, with $x = 5\cos(\frac{\pi}{30}t),\ y = 5\sin(\frac{\pi}{30}t)$, we have $s = \frac{\pi}{6}t$, so we obtain $t = \frac{6}{\pi}s$. Since $\frac{\pi}{30}t = \frac{\pi}{30}\frac{6}{\pi}s = \frac{1}{5}s$, so we get the parameterization $5\cos(\frac{1}{5}s),\ y = 5\sin(\frac{1}{5}s)$.

Note that regardless of which parameterization we started with, we obtained the *same* parameterization with respect to arc length. Whereas there are infinitely many ways to parameterize a curve with respect to time t, there is a *unique* way of parameterizing the curve with respect to arc length s (assuming we have settled upon a starting point and a forward direction for the curve). Thus, for any given curve, parameterization with respect to arc length is the most “natural” or “objective” parameterization.

If $v(t) = 1$ for all t, then $s(t) = \int_\alpha^t 1du == [u]_\alpha^t = t - \alpha$. Thus, $s = t - \alpha$, and $t = s + \alpha$.

For instance, suppose the unit circle is parameterized as $x = \cos t,\ y = \sin t$. So $v(t) = 1$.

- If we choose $\alpha = 0$ as our starting point, then $s = t$, and the parameterization with respect to arc length is $x = \cos s,\ y = \sin s$. When $s = 0$, we are at the point $(1, 0)$.
- If we choose $\alpha = \frac{\pi}{4}$ as our starting point, then $s = t - \frac{\pi}{4}$ and $t = s + \frac{\pi}{4}$, so the parameterization with respect to arc length is $x = \cos(s + \frac{\pi}{4}),\ y = \sin(s + \frac{\pi}{4})$. When $s = 0$, we are at the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$.

7. Three-Dimensional Motion

The above discussion dealt with motion in two-dimensional space. All our results can easily be adapted to three-dimensional space.

Suppose the motion of a particle is represented by the position function $\mathbf{r}(t) = < x(t), y(t), z(t) >$. The velocity is $\mathbf{v}(t) = < x'(t), y'(t), z'(t) >$ and the speed is $v(t) = |\mathbf{v}(t)| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$.

The differential of arc length is $ds = \sqrt{dx^2 + dy^2 + dz^2}$, where $dx = x'(t)dt,\ dy = y'(t)dt,$ and $dz = z'(t)dt.$ Note that $dx^2 + dy^2 + dz^2 = ds^2.$

Let α and β be two values of $t,$ with $\alpha < \beta.$ Let P be the point $(x(\alpha), y(\alpha), z(\alpha))$ and let K be the point $(x(\beta), y(\beta), z(\beta)).$ As time moves forward from α to $\beta,$ our particle moves along some curve from P to $K.$ The distance it travels (in other words, the length of the arc of the curve from P to K) is $\int_\alpha^\beta ds = \int_\alpha^\beta \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\,dt = \int_\alpha^\beta v(t)dt.$

Using P as our starting point, the arc length function is $s(t) = \int_\alpha^t \sqrt{x'(u)^2 + y'(u)^2 + z'(u)^2}\,du = \int_\alpha^t v(u)du.$ Note that $s'(t) = v(t).$

The parameterization $x = x(s),\ y = y(s),\ z = z(s)$ is known as the parameterization with respect to arc length.

If $v(t) = 1$ for all $t,$ then $s = t - \alpha,$ and $t = s + \alpha.$

Consider the helix $\mathbf{r}(t) = < 2\cos t, 2\sin t, 4t >.$ $\mathbf{v}(t) = < -2\sin t, 2\cos t, 4 >,$ so $v(t) = 2\sqrt{5}$ for all $t.$ For $\alpha = 0,$ the arc length function is $s(t) = \int_0^t 2\sqrt{5}\,du = 2\sqrt{5}\,t.$ Since $t = \frac{\sqrt{5}}{10}s,$ the parameterization with respect to arc length is $\mathbf{r}(s) = < 2\cos(\frac{\sqrt{5}}{10}s), 2\sin(\frac{\sqrt{5}}{10}s), \frac{2\sqrt{5}}{5}s >.$